Our Own Problems and Solutions to Accompany Topic 5

1. What is the monthly payment on a $275,000 mortgage loan with a 5.4% stated annual percentage rate (APR) of interest, if payments are to be made at the end of each month for 30 years?

Recall that a fixed-rate, fixed-payment mortgage loan is a present value of a level ordinary annuity application, and that we can always set up an annuity problem with the formula

\[ \text{PMT} \times \text{FAC} = \text{TOT} \]

With this formula in place, we can simply plug in the "knowns" and solve for the "unknowns:"

Because the $275,000 lump sum "Tot" exists intact in the present, we have a present value of an annuity problem (and with equal payments at the end of each period we have a level ordinary annuity). With a 5.4% stated annual percentage rate (APR) of interest and monthly payments, our monthly periodic interest rate is \(0.054 / 12 = 0.0045\), or .45%. With 30 years of monthly payments, we have \(30 \times 12 = 360\) payment periods. Therefore, with $275,000 borrowed we compute

\[
\text{PMT} \left(1 - \left(\frac{1}{1 + 0.0045}\right)^{360}\right) = 275,000
\]

\[
\text{PMT} \times 178.084624 = 275,000
\]

So \(\text{PMT} = 275,000 / 178.084624 = 1,544.21\).

Note that when computing loan payments we typically replace the present value of an annuity factor (here, 113.950820) with its reciprocal, the loan payment factor, so that we can multiply the amount borrowed by this factor to compute the periodic payment. \(1 / 178.084624 = 0.005615\), so

\[
\text{PMT} = 275,000 \times 0.005615 = 1,544.21
\]

On a Texas Instruments BA II Plus financial calculator (other brands/models may follow slightly different key sequences), set payments per period to 1 instead of the default setting of 12. Type in $275,000 +/- PV, $0 FV, 360 N, 5.4 / 12 = I/Y (the periodic interest rate is 5.4% / 12 = .45%, or .0045); CPT PMT. It should show $1,544.21, just as we computed manually above.

2. What is the monthly payment on a $190,000 mortgage loan with a 5.725% effective annual rate (EAR) of interest, if payments are to be made at the end of each month for 25 years? What if they are to be made bi-weekly for 25 years (our theoretical case with actual bi-weekly payments, not the “bi-weekly” loans generally available).

This problem is quite similar to the previous problem; the only substantive difference is that here we are given the EAR as the annual interest rate measure that we must convert into a monthly "working" rate before computing. Whereas the APR is a simple annual rate measure (if payments are made monthly then the APR is just 12 times the monthly rate, such that we find the monthly rate by dividing the APR by 12), the EAR is a more complicated annual rate measure - it includes the impact of compounding over the course of a year. If payments are made monthly the EAR is the result of compounding the monthly rate 12 times, so to compute a monthly "working" rate we must "un-compound" 12 times, by taking the twelfth root of (1 + EAR).
Again we have a lump sum "TOT" ($190,000) that exists intact in the present, so this problem is again a present value of an annuity situation (and with equal payments at the end of each period we have a level ordinary annuity). But now the annual interest rate measure - we always talk about rates in annual terms, but we must work with a rate that corresponds to the timing of the payments and compounding - is a compounded EAR. With that EAR given as 5.725%, and with monthly payments, our monthly periodic interest rate is $\left(\frac{12}{1.05725} - 1\right) = .00465$, or .465%. With 25 years of monthly payments, we have 25 x 12 = 300 payment periods. Therefore, with $190,000 borrowed we compute

$$\text{PMT} \left( \frac{1 - \left( \frac{1}{1.00465} \right)^{300}}{.00465} \right) = \$190,000$$

$$\text{PMT} \times 161.583728 = \$190,000$$

So $\text{PMT} = \$190,000 \div 161.583728 = \$1175.86 \ OR \ \text{PMT} = \$190,000 \times .006189 = \$1175.86$.

On a Texas Instruments BA II Plus financial calculator (other brands/models may follow slightly different key sequences), set payments per period to 1 instead of the default setting of 12. Type in $190,000 +/- \ PV, \ 0 \ FV, \ 300 \ N, \ 1.05725 \ y^x (1 \div 12) = -1 \times 100 = I/Y \ (the \ periodic \ interest \ rate \ is \ the \ rate \ that \ compounds \ 12 \ times \ to \ be \ 5.725\%, \ which \ is .465\%, \ or \ .00465); \ CPT \ \text{PMT}. \ \text{It \ should \ show} \ \$1175.87 \ [a \ tiny \ rounding \ difference \ from \ the \ answer \ computed \ above, \ because \ (1.00465)^{12} -1 \ is \ not \ exactly \ 5.725\%; \ it \ is \ actually \ 5.7249438\%].$

Ordinarily we are not converting an APR to an EAR or vice versa; more typically we convert either an APR or an EAR (an annual rate that we talk about) into a periodic r that we actually work with. Entering the values into the calculator is a bit more cumbersome when the annual figure we know is the EAR rather than the APR. The good news is that in real world loan situations we are likely to be told the APR rather than the EAR. Federal law requires only that the lender report the APR to the borrower; why would the lender want to report the higher EAR as the interest cost?

If payments are made bi-weekly (every other week, or 26 times during the year) the EAR is the result of compounding the monthly rate 26 times, so to compute a monthly "working" r we must "un-compound" 26 times, by taking the twenty-sixth root of (1 + EAR): $\sqrt[26]{1.05725} - 1 = .00214349, or .214349\%$. If there are 25 years of bi-weekly payments we have 25 x 26 = 650 payment periods. With $190,000 borrowed we compute

$$\text{PMT} \left( \frac{1 - \left( \frac{1}{1.00214349} \right)^{650}}{.00214349} \right) = \$190,000$$

$$\text{PMT} \times 350.534068 = \$190,000$$

So $\text{PMT} = \$190,000 \div 350.534068 = \$542.03 \ OR \ \text{PMT} = \$190,000 \times .002853 = \$542.03$.

On a Texas Instruments BA II Plus financial calculator (other brands/models may follow slightly different key sequences), set payments per period to 1 instead of the default setting of 12. Type
in $190,000 +/− PV, $0 FV, 26 × 25 = (shows 650) N, 1.05725 y− (1 ÷ 26) = - 1 × 100 = I/Y (the periodic interest rate is the rate that compounds 26 times to be 5.725%, which is .00214349, or .00214349); CPT PMT. It should show $542.03.

Finally, note that the annual percentage rate (APR) in each of the two situations above would be .00465 × 12 = .055801 or 5.5801% (vs. the higher 5.725% EAR) for the monthly payment case and .00214349 × 26 = .05573084 or 5.573084% (vs. the higher 5.725% EAR) for the bi-weekly payment case. If given this simpler APR we would have found the periodic r for our PV of annuity factor just by computing .00465 ÷ 12 = .000395833 or .00214349; CPT PMT. It should show $542.03.

3. If someone can afford to make a monthly principal-plus-interest payment of $1,550, and if a local lending institution is willing to lend at a 6.72% annual percentage rate (APR) of interest with monthly payments over 15 years, how much can the individual afford to borrow?

Here we use the same skeleton as in the previous problem, but now it is the loan principal “TOT” that we solve for. Again we plug in the “knowns” and solve for the “unknowns.” Again the lump sum “TOT” (here the unknown we are solving for) exists intact in the present, so again we have a present value of a level ordinary annuity problem (as always is true in dealing with fixed-payment mortgage loan computations). With a 6.72% stated annual percentage rate (APR) of interest and monthly payments, our monthly periodic interest rate is .0672 ÷ 12 = .0056. With 15 years of monthly payments, we have 15 × 12 = 180 payment periods. Therefore, with a $1,550 monthly payment we compute

\[
1,550 \left( \frac{1 - \left( \frac{1}{1.0056} \right)^{180}}{.0056} \right) = \text{TOT}
\]

\[
1,550 \times 113.218609 = $175,488.84 .
\]

Let’s say that the home buyer borrows this amount and makes a 20% down-payment. How expensive a house can he afford to buy? If 80% of the price is the $175,488.84 borrowed, then the total he can pay is $175,488.84 ÷ .80 = $219,361.05 , with 20% ($43,872.21) out-of-pocket and 80% ($175,488.84) borrowed. (Do you see that the annual interest rate could have been given as a (1.056)^12− 1 = 6.930889% EAR rather than the 6.72% APR?)

On a financial calculator, type in $1,550 PMT, $0 FV, 180 N, 6.72 ÷ 12 = I/Y (the periodic interest rate is 6.72% ÷ 12 = .56%, or .0056); CPT PV. It should show $-175,488.84 , just as we computed manually above. (It shows as negative because that is what comes out of the lender's pocket, but we could as easily have put a negative sign on the $1,550 payment and gotten a positive $175,488.84 present value. The logic of the financial calculator is that one of the dollar values must be a negative. If you do not enter a negative dollar value and you are solving for a dollar value the calculator will merely assign a negative value to the answer. If you do not enter a negative dollar value and you are solving for n or r the calculator will show an error message.)

4. A $157,000 loan has a 5.82% APR interest rate and a 30-year amortization period (monthly payments). If the borrower sells the house and repays the remaining principal balance after 12 years, how much in principal is still owed at that date (and how much has been repaid)? Is the amount repaid approximately 12/30 = 40% of the original
principal? A $234,000 loan with a 4.83% APR has a 15-year amortization period (monthly payments); if the house is sold after 9 years and 5 months, how much in principal does the borrower still owe/how much has been repaid?

Here we compute the “balloon payment” that would be owed if a loan were to be retired prior to the original maturity date. If the interest rate is stated as a 5.82% APR, the accompanying monthly rate we compute with is .0582 ÷12 = .00485, and the monthly payment is computed as

$$PMT \left(1 - \frac{1}{1.00485^{360}}\right) = 157,000$$

$$PMT \times 70.060139 = 157,000$$

So $PMT = 157,000 ÷ 70.060139 \text{ OR } PMT = 157,000 \times .005880 = 923.20$

Then we can compute the remaining principal owed as the present value of the remaining stream of payments. The amount owed on a loan at any time is the present value of the remaining payment stream; this extremely important lesson will serve us in computations throughout the semester. Note, for example, that at this loan’s outset the amount owed is the PV of 360 payments of $923.20 each:

$$923.20 \left(1 - \frac{1}{1.00485^{360}}\right) = 157,000 \checkmark$$

12 years into the 30-year amortization period (with 18 years = 216 months of payments remaining), the amount still owed is

$$923.20 \left(1 - \frac{1}{1.00485^{216}}\right) =$$

$$923.20 (133.676703) = 123,410.71$$

(so the amount that has been repaid is $157,000 - 123,410.71 = 33,589.29$).

Another approach is to use the formula for the proportion of a loan that has been repaid:

$$\frac{(1 + \text{Periodic Rate})^{\text{Short Period} - 1}}{(1 + \text{Periodic Rate})^{\text{Long Period} - 1}}$$

such that

$$1 - \frac{(1 + \text{Periodic Rate})^{\text{Short Period} - 1}}{(1 + \text{Periodic Rate})^{\text{Long Period} - 1}}$$

is the proportion of the original principal that has not been repaid by a specified date, and thus is still owed. The beauty of this approach - which, as one of our handouts explains, is just a shorter form of the PV of remaining payments approach used above - is that we do not have to know the monthly payment to compute what is still owed; the computation is based solely on original principal and proportion repaid. Here, 144 months into the loan's 360-month original life, we compute

$$157,000 \left[1 - \frac{(1.00485)^{144} - 1}{(1.00485)^{360} - 1}\right] = 157,000 (1 - .213945) = 157,000 (.786055) = 123,410.71$$
(still owed, with $157,000 - $123,410.71 = $33,589.29 already repaid).

Another way to compute is with a financial calculator, though your aging instructor fears that people who simply learn to hit buttons on a financial calculator may not truly understand the underlying ideas. But for you financial calculator jocks: enter $157,000 +/- PV, $0 FV, 360 N, 5.82 ÷ 12 = I/Y (the periodic interest rate is 5.82% ÷ 12 = .485%, or .00485); CPT PMT (should show $923.20. Then to see what is still owed after 144 months enter 144 N; CPT FV (should show $123,410.71; then enter +/- and + $157,000 = to find the $33,589.29 that has been repaid).

Another, more versatile financial calculator approach is to use the AMORT function. First, you need to have the original loan information in place, so re-enter $0 FV and 360 N. Then to see what has happened during months 1 - 144 type 2nd AMORT 1 ENTER ↓ 144 ENTER ↓ (it should show $123,410.71 as the principal balance still owed 144 months into the loan's 360-month life), then ↓ again (it should show $33,589.29 as principal repaid over the first 144 months), and then ↓ again (it should show $99,351.91 as the total interest paid over those 144 months - not asked for in this problem, but a value that can be useful to know in some situations).

Here we are 12 years, which is 40%, of the way through a 30 year amortization period, but all that has been repaid is $33,589.29 ÷ $157,000 = 21.3945% of the principal borrowed. Why? Recall that each payment in the early years is largely interest, with only a small amount of principal repaid -- interest can be charged only on principal that remains outstanding. So 40% of the way into the repayment period nowhere near 40% of the principal has been repaid.

If a $234,000 loan has a 15-year/180-month amortization and an interest rate stated as a 4.83% APR, the monthly rate we compute with is .0483 ÷12 = .004025, and the monthly payment is computed as

$$\text{PMT} \left( \frac{1 - \left( \frac{1}{1.004025} \right)^{180}}{.004025} \right) = \$234,000$$

$$\text{PMT} \times 127.882748 = \$234,000$$

So $\text{PMT} = \$234,000 ÷ 127.882748 \ OR \ \text{PMT} = \$234,000 \times 0.007820 = \$1,829.80$.

Then 9 years and 5 months (113 months) into the 15 year (180 month) amortization period (with 180 - 113 = 67 months of payments remaining), the amount still owed (a partial year causes no complications; our computations involve months, with yearly measures simply being the "talked about" figures that must be converted to monthly figures before we compute) is

$$\$1,829.80 \left( \frac{1}{1.004025} \right)^{67} \ OR \ \$234,000 \left( 1 - \frac{(1.004025)^{113} - 1}{(1.004025)^{180} - 1} \right) = \$107,269.05$$

(so the amount that has been repaid is $234,000 - $107,269.05 = $126,730.95 - just over half has been repaid even though we are 9/15 = 60% into the amortization period).
Financial calculator: enter $234,000 +/- PV, $0 FV, 180 N, 4.83 ÷ 12 = I/Y (the periodic interest rate is 4.83% ÷ 12 = .4025%, or .004025); CPT PMT (should show $1,829.80). To determine what is still owed after 113 months enter 113 N; CPT FV (should show $107,269.05; then enter +/- and + $234,000 = to find the $126,730.95 that has been repaid). Or with the AMORT function, re-enter $0 FV and 180 N so you have the original loan information in place. Then to determine where things stand after 113 months type 2nd AMORT 1 ENTER ↓ 113 ENTER ↓ (it should show $107,269.05 as the principal still owed 113 months into the loan’s 180-month life), then ↓ again (it should show $126,730.95 as principal repaid over the first 113 months), then ↓ again (should show $80,036.58 in total interest paid over those 113 months).

5. A borrower obtains a $295,000 mortgage loan with a 6.3% stated annual percentage rate (APR) of interest and monthly payments to be made over 30 years. Question a: How much is each monthly payment?

With a 6.3% stated annual percentage rate (APR) of interest and monthly payments, our monthly periodic interest rate is .063 ÷ 12 = .00525. With 30 years of monthly payments, we have 30 x 12 = 360 payment periods. With $295,000 borrowed we compute

\[ \text{PMT} \left( 1 - \left( \frac{1}{1.00525} \right)^{360} \right) = \frac{295,000}{.00525} \]

\[ \text{PMT} \times 161.557990 = 295,000 \]

So \( \text{PMT} = \frac{295,000}{161.557990} \) OR \( \text{PMT} = 295,000 \times .006190 = 1,825.97 \).

On a financial calculator, type in $295,000 +/- PV, $0 FV, 360 N, 6.3 ÷ 12 = I/Y (the periodic interest rate is 6.3% ÷ 12 = .525%, or .00525); CPT PMT. It should show $1,825.97, just as we computed manually above.

b. How much interest does the borrower pay during the first month of the loan’s life?

This part is easy. Interest paid in any period is just the periodic (here monthly) rate times the amount of principal owed at the start of the period. In the first month the entire amount of principal borrowed is still owed, so the interest component of the first month's payment is just

\[ .00525 \times 295,000 = 1,548.75 \, \text{.} \]

Recall that the total payment, which consists of both interest on the unpaid principal balance and some repayment of principal, is $1,825.97. So the first month’s payment of $1,825.97 consists of $1,548.75 in interest and a much smaller ($1,825.97 - $1,548.75) = $277.22 in principal repaid.

Indeed, note that the loan payment factor (here .006190) must be enough to pay the period’s interest and also retire some principal. Specifically, it is the sum of the .00525 interest rate plus the sinking fund factor (the reciprocal of the future value of an annuity factor, which accounts for paying back principal systematically over the amortization period). The sinking fund factor in this case is
\[
\left( \frac{.00525}{(1.00525)^{360}-1} \right) = .000940
\]

such that the sum of the monthly periodic interest rate and the sinking fund factor is \( .00525 + .000940 = .006190 \), the loan payment factor.

c. How much interest does the borrower pay during the first year of the loan’s life?

This part is harder. First we might consider the "brute force" approach:

<table>
<thead>
<tr>
<th>Month</th>
<th>Initial Balance</th>
<th>.063 \div 12 =</th>
<th>Adjusted Balance</th>
<th>Minus Payment Balance</th>
<th>Ending Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$295,000.00</td>
<td>$ 1,548.75</td>
<td>$296,548.75</td>
<td>$1,825.97</td>
<td>$294,722.78</td>
</tr>
<tr>
<td>2</td>
<td>$294,722.78</td>
<td>$ 1,547.29</td>
<td>$296,270.07</td>
<td>$1,825.97</td>
<td>$294,444.11</td>
</tr>
<tr>
<td>3</td>
<td>$294,444.11</td>
<td>$ 1,545.83</td>
<td>$295,989.94</td>
<td>$1,825.97</td>
<td>$294,163.97</td>
</tr>
<tr>
<td>4</td>
<td>$294,163.97</td>
<td>$ 1,544.36</td>
<td>$295,708.33</td>
<td>$1,825.97</td>
<td>$293,882.36</td>
</tr>
<tr>
<td>5</td>
<td>$293,882.36</td>
<td>$ 1,542.88</td>
<td>$295,425.24</td>
<td>$1,825.97</td>
<td>$293,599.27</td>
</tr>
<tr>
<td>6</td>
<td>$293,599.27</td>
<td>$ 1,541.40</td>
<td>$295,140.67</td>
<td>$1,825.97</td>
<td>$293,314.70</td>
</tr>
<tr>
<td>7</td>
<td>$293,314.70</td>
<td>$ 1,539.90</td>
<td>$294,854.60</td>
<td>$1,825.97</td>
<td>$293,028.63</td>
</tr>
<tr>
<td>8</td>
<td>$293,028.63</td>
<td>$ 1,538.40</td>
<td>$294,567.0</td>
<td>$1,825.97</td>
<td>$292,741.06</td>
</tr>
<tr>
<td>9</td>
<td>$292,741.06</td>
<td>$ 1,536.89</td>
<td>$294,277.95</td>
<td>$1,825.97</td>
<td>$292,451.98</td>
</tr>
<tr>
<td>10</td>
<td>$292,451.98</td>
<td>$ 1,535.37</td>
<td>$293,987.35</td>
<td>$1,825.97</td>
<td>$292,161.38</td>
</tr>
<tr>
<td>11</td>
<td>$292,161.38</td>
<td>$ 1,533.85</td>
<td>$293,695.23</td>
<td>$1,825.97</td>
<td>$291,869.26</td>
</tr>
<tr>
<td>12</td>
<td>$291,869.26</td>
<td>$ 1,532.31</td>
<td>$293,401.58</td>
<td>$1,825.97</td>
<td>$291,575.61</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>\textbf{$18,487.24}</td>
</tr>
</tbody>
</table>

But this "brute force" approach, which shows the loan’s month-by-month amortization over the first year, is a pain if you don’t have a spreadsheet, especially if you are dealing with a longer period (think of doing this for 10 years, or 120 months!). A more direct approach is to recall that the remaining principal balance, at any point during the life of a long-term loan, is the present value of the remaining payment stream when discounted at the original contract’s periodic interest rate. Here, at the end of year 1, with 29 years (348 months) of payments left to make, the present value of the remaining payment stream is

\[
$1,825.97 \left( 1 - \frac{1}{(1.00525)^{348}} \right) = \text{TOT}
\]

\[
$1,825.97 \times 159.682606 = \$291,575.61\,.
\]

If that amount of the original $295,000 principal remains unpaid, then the remaining $295,000 - $291,575.61 = $3,424.39 has been repaid. Everything else paid during year 1 has been interest. So here we note that

\[
\text{Total payments during year } 1: 12 \times \$1,825.97 = \$21,911.64
\]

\[
\text{Minus principal portion} \quad 3,424.39
\]

\[
\text{Equals interest paid during year } 1 \quad \$18,487.24\,.
\]
Yet a third way to find the answer is with our formula for computing the proportion of principal that has been repaid (useful because we would not have to first compute the monthly payment):

\[
\left( \frac{(1 + \text{Periodic Rate})^{\text{Short Period}} - 1}{(1 + \text{Periodic Rate})^{\text{Long Period}} - 1} \right)
\]

(such that 1 minus that proportion is the proportion of principal that remains to be paid). In this case, 12 months (short period) into the loan's 360-month (long period) life, we compute

\[
\left( \frac{(1.00525)^{12} - 1}{(1.00525)^{360} - 1} \right) = \frac{0.064851}{5.586724} = 0.011608 .
\]

That small decimal fraction is the proportion of the original loan principal that has been repaid by the time the "short period" has ended (here, 12 months into the 360-month life). Everything else paid during that "short period" was interest. So here we note that

\[
\begin{align*}
\text{Total payments during year 1:} & \quad 12 \times 1,825.97 = 21,911.64 \\
\text{Minus principal portion:} & \quad 0.011608 \times 295,000 = 3,424.39 \\
\text{Equals interest paid during year 1:} & \quad 18,487.24
\end{align*}
\]

(you could end up with some slight rounding differences if you are rounding to whole cents rather than keeping your answers in memory as you proceed).

And yes, a financial calculator is useful here as well, but your aging instructor fears that people who simply learn to hit buttons on a financial calculator may not truly understand the underlying ideas. But for you financial calculator jocks, you should already have entered $295,000 +/- PV, $0 FV, 360 N, 6.3 \div 12 = I/Y (the periodic interest rate is 6.3\% \div 12 = .525\%, or .00525); CPT PMT (and gotten $1,825.97). Now, to figure out what has happened during months 1 - 12, type 2nd AMORT 1 ENTER \downarrow 12 ENTER \downarrow (it should show $291,575.61 as the principal balance still owed 12 months into the loan's 360-month life), then \downarrow again (it should show $3,424.39 as principal repaid over the first 12 months), and then \downarrow again (it should show $18,487.24 as the interest paid over those 12 months).

d. If the borrower makes regular monthly payments for four years and then decides to pay the loan off, how much principal is still owed at the end of year 4? How much in total interest does the borrower pay over this four-year period?

Four years into the loan's life, with 26 years (312 months) of payments to go, the remaining outstanding principal balance is the present value of the remaining payment stream:

\[
$1,825.97 \left( \frac{1 - \left( \frac{1}{1.00525} \right)^{312}}{.00525} \right) = \text{TOT}
\]

\[
$1,825.97 \times 153.294667 = $279,911.42
\]

Alternatively, 4 years or 48 months (short period) into the loan's 360-month (long period) life, we compute the proportion of the original principal that has been repaid as
\[
\left( \frac{(1.00525)^{48} - 1}{(1.00525)^{360} - 1} \right) = \left( \frac{.285748}{5.586724} \right) = .051148.
\]

With that proportion of the original principal repaid, 100% minus that proportion, or 1 - .051148 = .948852 is the proportion remaining unpaid. So after 36 months he still owes .948852 x $295,000 = $279,911.42. This is the "balloon payment" that the borrower would have to make to retire the loan after four years, at the end of month 48. On a financial calculator, with all the information from above still entered (or re-enter it if need be), hit 48 N (to change our time focus from 360 months to 36); CPT FV, and it should show $279,911.42.

Then with $279,911.42 still owed, the principal that has been repaid over the first 48 months is $295,000 - $279,911.42 = $15,088.58. So we can show

<table>
<thead>
<tr>
<th>Description</th>
<th>Amount</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total payments during years 1 - 4:</td>
<td>$87,646.55</td>
</tr>
<tr>
<td>Minus principal portion</td>
<td>$15,088.58</td>
</tr>
<tr>
<td>Equals interest paid during first 4 years</td>
<td>$72,557.97</td>
</tr>
</tbody>
</table>

The harder, but perhaps more versatile, way to deal with this situation is to type 2nd AMORT 1 ENTER ↓ 48 ENTER ↓ (it should show $279,911.42 as the principal balance still owed 48 months into the loan's life), then ↓ again (it should show $15,088.58 as principal repaid over those 48 months), then ↓ again (it should show $72,557.97 as the interest paid over those 48 months).

6. What annual percentage rate (APR) of interest is the lender charging on a $168,000 mortgage loan with payments of $1,208.91 every month for 25 years? What is the effective annual rate (EAR) of interest?

Here we use the same skeleton as before, but now the interest rate charged on the loan is the unknown to solve for. Because the payments occur monthly, we should think in terms of having 25 x 12 = 300 monthly payment periods and a periodic interest rate of \( r \):

\[
PMT \times FAC = TOT
\]

\[
$1,208.91 \left( 1 - \left( \frac{1}{1 + r} \right)^{300} \right) = $168,000
\]

Because the present value of a level ordinary annuity factor's denominator contains \( r \) and its numerator contains \( r \) to a power (here \( r^{300} \)), there is no way to isolate \( r \) on one side of the equals sign and have only terms without \( r \) on the other side. Thus there is no way to directly solve for \( r \); we must use trial and error. We would find the monthly periodic interest rate \( r \) to be .006; let's double-check:

\[
$1,208.91 \left( 1 - \left( \frac{1}{1.006} \right)^{300} \right) = $168,000
\]

\[
$1,208.91 \times 138.968276 = $168,000. \checkmark
\]
Now to solve what the question asks for: with a monthly periodic rate of .006, we have an annual percentage rate (APR) of .006 \times 12 = .072, or 7.2%, while the EAR would be \((1.006)^{12} - 1 = .074424\), or 7.4424%.

On a financial calculator, which is programmed to handle the trial and error attempts quickly, we would enter \$168,000 +/- PV, $0 FV, $1,208.91 PMT, 300 N; CPT I/Y. It should show .600001; that is a monthly rate because the $1,208.91 payment and 300 time periods are monthly figures. (The tiny rounding difference arises because the monthly payment for a $168,000 loan with a .6% monthly interest rate is technically $1,208.909002 rather than the rounded $1,208.91 we have used.) Then multiply the calculator solution by 12: \((.6 + 100) \times 12 = .072\) to get the APR, or take \((1 + r)\) to the 12th power: \((1.006)^{12} - 1 = .074424\) to compute the EAR.

Caution with your calculator: watch the negative signs when solving for the rate (I/Y, r, i, or however the calculator designates it). In a loan situation the lender gives up money up front and then receives money as payments are made (borrower is in just the opposite position). So either the PV or the PMT has to be shown as a negative or the calculator can’t solve for r, because if you’re a lender and you get money up front and then continue to get payments later you’re getting an infinite rate of return, which the calculator can’t find as a possible solution. It’s not a problem when you are solving for the payment (or the loan amount) and forget to enter the loan amount (or the payment) as a negative, because the calculator will simply assign the negative value to whatever it computes. But if you are solving for the rate of return or number of time periods and do not enter a negative dollar amount you will get an error message.

7. You want to buy a house with a $260,000 asking price. You plan to make a 20% out-of-pocket down-payment ($52,000) and then borrow the other 80% ($208,000). After completing its underwriting activities, the bank where you applied for a loan has good news and bad news. The good news is that it has approved you for a 30-year loan (monthly payments) with a 6.36% annual percentage rate (APR) of interest. The bad news is that the bank’s appraiser feels the house is worth only 92% of the asking price, and the bank will lend only 80% of the appraised value. Question a: How much will the monthly payment be?

The bank feels that the home is worth \(.92 \times \$260,000 = \$239,200\). It will lend you 80% of that value, or \(.8 \times \$239,200 = \$191,360\). (Therefore if the home’s seller insists on charging a $260,000 price, you will have to make a larger down-payment than the $52,000 you had planned: $260,000 paid - $191,360 borrowed = $68,640 down-payment.) Here we have payments over 30 years or 360 monthly periods, and a monthly periodic interest rate of \(0.0636 \div 12 = .0053\). Because computing equal loan payments is a PV of a level ordinary annuity application, we can set the problem up as

\[
P_{MT} \left( \frac{1 - \left( \frac{1}{1.0053} \right)^{360}}{.0053} \right) = \$191,360
\]

\[
P_{MT} \times 160.542205 = \$191,360
\]

So \(P_{MT} = \$191,360 \div 160.542205 = \$1,191.96\)  OR \(P_{MT} = \$191,360 \times .006229 = \$1,191.96\).

On a financial calculator, we would enter \$191,360 +/- PV, $0 FV, 360 N, 6.36 \div 12 = I/Y (the periodic interest rate is 6.36% \div 12 = .53\% or .0053); CPT PMT. It should show $1,191.96, just as we computed manually above.
b. What amount of principal will still be owed at the end of year 3?

Three years into the loan's life, with 27 years (324 months) of payments to go, the remaining outstanding principal balance is the present value of the remaining payment stream:

$$1,191.96 \left( \frac{1 - (\frac{1}{1.0053})^{324}}{.0053} \right) = TOT$$

$$1,191.96 \times 154.644463 = $184,330.12 .$$

Alternatively, 3 years or 36 months (short period) into the loan's 360-month (long period) life, we compute the proportion of the original principal that has been repaid as

$$\left( \frac{(1.0053)^{36} - 1}{(1.0053)^{360} - 1} \right) = \frac{.209608}{5.705725} = .036736 .$$

With that proportion of the original principal repaid, 100% minus that proportion, or 1 - .036736 = .963264 is the proportion remaining unpaid. So after 36 months he still owes .963264 x $191,360 = $184,330.12. This is the "balloon payment" that the borrower would have to make to retire the loan after three years, at the end of month 36.

On a financial calculator, you can proceed in a couple of ways. The easier is as follows: with all the information from above still entered (or re-enter it if need be), hit 36 N (to change our time focus from 360 months to month 36); CPT FV, and it should show $184,330.12. The harder, but perhaps more versatile, way is to type 2nd AMORT 1 ENTER ↓ 36 ENTER ↓ (it should show $184,330.12 as the principal balance still owed 36 months into the loan's life). Go ahead and hit ↓ again (it should show $7,029.88 as principal repaid over those 36 months), and then ↓ again (it should show $35,880.71 as the interest paid over those 36 months).

c. Set up a three-year amortization schedule showing the total amount paid annually, broken down into each year’s total of principal and interest and the remaining principal balance at the end of each year. Do not make the problem harder than it needs to be: you do not have to complete a monthly amortization over 36 months. But do not make the problem easier than it is; you can not compute yearly totals simply by treating the payments as if they were made annually rather than monthly.

We might begin by noting that the payments made each year total 12 x $1,191.96 = $14,303.53 . Next we can compute the remaining principal balance at the end of each of the first three years of the loan's life. The difference between the remaining principal balance at the end of year \( t - 1 \) and the remaining principal balance at the end of year \( t \) is the amount of principal repaid during year \( t \):

Yr 0 (today): No principal has yet been repaid, so you still owe the full $191,360.00

Yr 1: \( 1,191.96 \left( \frac{1 - (\frac{1}{1.0053})^{348}}{.0053} \right) \) OR \$191,360 \left[ 1 - \left( \frac{(1.0053)^{12} - 1}{(1.0053)^{360} - 1} \right) \right] = $189,163.68 $2,196.32

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Yr 2: $1,191.96 \left(1 - \frac{1}{1.0053^{336}}\right)$ OR $191,360 \left[1 - \frac{(1.0053)^{24}-1}{(1.0053)^{360}-1}\right] = $186,823.53 $2,340.15

Yr 3: $1,191.96 \left(1 - \frac{1}{1.0053^{324}}\right)$ OR $191,360 \left[1 - \frac{(1.0053)^{36}-1}{(1.0053)^{360}-1}\right] = $184,330.12 $2,493.41

We can then set up a yearly amortization schedule, based on each year's total monthly payments (interest paid each year is simply the difference between the total payments and the principal):

<table>
<thead>
<tr>
<th>Year</th>
<th>Beginning Balance</th>
<th>Total Payments</th>
<th>Principal</th>
<th>Interest</th>
<th>Ending Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$191,360.00</td>
<td>$14,303.53</td>
<td>$2,196.32</td>
<td>$12,107.21</td>
<td>$189,163.68</td>
</tr>
<tr>
<td>2</td>
<td>$189,163.68</td>
<td>$14,303.53</td>
<td>$2,340.15</td>
<td>$11,963.38</td>
<td>$186,823.53</td>
</tr>
<tr>
<td>3</td>
<td>$186,823.53</td>
<td>$14,303.53</td>
<td>$2,493.41</td>
<td>$11,810.12</td>
<td>$184,330.12</td>
</tr>
</tbody>
</table>

On a financial calculator, we can quickly compute what happens year to year. Make sure everything is entered as you needed it for computing the payment in part a (or re-enter it). The ending balance for each year is most easily found as the FV at the targeted point in time, but that approach does not give us everything we would like for the amortization schedule. Instead type 2nd AMORT 1 ENTER ↓ 12 ENTER ↓ (it should show $189,163.68 as the principal balance still owed 12 months into the loan's life), then ↓ again (it should show $2,196.32 in principal paid in year 1), and ↓ again (it should show $12,107.21 in interest paid in year 1). Then type 2nd AMORT 13 ENTER ↓ 24 ENTER ↓ (it should show $186,823.53 as the principal balance still owed 24 months into the loan's life), then ↓ again (it should show $2,340.15 in principal paid in year 2), and ↓ again (it should show $11,963.38 in interest paid in year 2). Then type 2nd AMORT 25 ENTER ↓ 36 ENTER ↓ (should show $184,330.12 as the principal balance still owed 36 months into the loan's life), then ↓ again (should show $2,493.40 in principal paid in year 3), and ↓ again (it should show $11,810.13 in interest paid in year 3). (There is a slight 1 cent rounding difference because in the year-by-year approach above we were subtracting based on whole cents, whereas the financial calculator solution keeps all values in memory to many decimal places.)

8. What will the monthly payment be on a $175,000 mortgage loan with a stated annual interest rate (APR) of 5.472% and a 20-year amortization period? What about a $175,000 loan with a 5.472% APR and a 30-year amortization period? Since 30-year period is 50% longer than the 20 year period, is the monthly payment on the 20-year loan 50% higher than that on the 30-year loan? Why?

Here we simply compute the monthly payment on a $175,000 fixed-rate, fixed-payment mortgage loan with a 5.472% ÷ 12 = .456% monthly interest rate with maturities of 240 months and 360 months:

$$PMT \left(1 - \frac{1}{1.00456^{240}}\right) = $175,000$$

$$PMT \times 145.707423 = $175,000$$

So $$PMT = $175,000 \div 145.707423 = $1,201.04$$ OR $$PMT = $175,000 \times .006863 = $1,201.04.$$
\[
P_{MT} \left( 1 - \left( \frac{1}{1.00456} \right)^{360} \right) = $175,000
\]

\[
P_{MT} \times 176.667993 = $175,000
\]

So \( P_{MT} = $175,000 \div 176.667993 = $990.56 \) \text{ OR } \( P_{MT} = $175,000 \times .005660 = $990.56 \).

With a financial calculator, enter $175,000 +/- PV, $0 FV, 240 N, 5.472 \div 12 = I/Y (the periodic interest rate is 5.472% \div 12 = .456\%, or .00456); CPT PMT. It should show $1,201.04, as above. Then enter 360 N and CPT PMT; it should show $990.56 as above. The 30 year amortization period is (30/20) \cdot 1 = 50\% greater than the 20 year amortization period, yet the 20-year payment is only $(1,201.04/$990.56) \cdot 1 = 21.25\% greater than the 30-year payment. Why? When money is paid back over a shorter time span, but at the same interest rate, principal is repaid more rapidly and thus far less in total interest is paid over the shorter amortization period.

9. Loan A has $212,500 in principal, monthly payments for 30 years, and a 6.384\% APR interest rate. Loan B has $212,500 in principal, monthly payments for 30 years, and a 6.924\% APR interest rate. An observer states, “Since bigger payments are made on the loan with the higher interest rate, the remaining principal balance will be the same for either loan as time passes.” Is this statement correct? Decide by computing the amount of principal still owed on each loan after 6 years, 17 years, and 28 years of payments will have been made.

The monthly payments on these two fixed-rate, fixed-payment mortgage loans ($212,500 in principal for both, 30 \times 12 = 360 monthly payments for both, and respective monthly interest rates of .06384 \div 12 = .00532 and .06924 \div 12 = .00577) are:

\[
P_{MT} \left( 1 - \left( \frac{1}{1.00532} \right)^{360} \right) = $212,500
\]

\[
P_{MT} \times 160.138705 = $212,500
\]

So \( P_{MT} = $212,500 \div 160.138705 = $1,326.97 \) \text{ OR } \( P_{MT} = $212,500 \times .006245 = $1,326.97 \).

\[
P_{MT} \left( 1 - \left( \frac{1}{1.00577} \right)^{360} \right) = $212,500
\]

\[
P_{MT} \times 151.467827 = $212,500
\]

So \( P_{MT} = $212,500 \div 151.467827 = $1,402.94 \) \text{ OR } \( P_{MT} = $212,500 \times .006620 = $1,402.94 \).

With a financial calculator, enter $212,500 +/- PV, $0 FV, 360 N, 6.384 \div 12 = I/Y (the periodic interest rate is 6.384\% \div 12 = .532\%, or .00532); CPT PMT. It should show $1,326.97, as above. Then enter 6.924 \div 12 = I/Y (the periodic interest rate is 6.924\% \div 12 = .577\%, or .00577); CPT PMT. It should show $1,402.94, as above. But actually we do not need to know these payments to
compute remaining principal balance at any given time; we could compute remaining balance knowing only the $212,500 original principal and the number of months elapsed with the equation

\[
1 - \left( \frac{(1 + \text{Periodic Rate})^{\text{Short Period}} - 1}{(1 + \text{Periodic Rate})^{\text{Long Period}} - 1} \right),
\]

which gives the proportion of a loan's principal that has been repaid by a particular month of the amortization period. Let's follow our earlier practice of computing both ways. Six years into the 30-year payment period (with 72 months gone and 288 remaining) the amounts owed are:

Loan A: $1,326.97 \left( \frac{1 - \left( \frac{1}{1.00532} \right)^{288}}{.00532} \right) \text{ OR } 212,500 \left[ 1 - \left( \frac{(1.00532)^{72} - 1}{(1.00532)^{360} - 1} \right) \right] = 195,317.76

Loan B: $1,402.94 \left( \frac{1 - \left( \frac{1}{1.00577} \right)^{288}}{.00577} \right) \text{ OR } 212,500 \left[ 1 - \left( \frac{(1.00577)^{72} - 1}{(1.00577)^{360} - 1} \right) \right] = 196,772.84

Seventeen years into the 30-year payment period (with 204 months gone and 156 remaining) the amounts owed are:

Loan A: $1,326.97 \left( \frac{1 - \left( \frac{1}{1.00532} \right)^{156}}{.00532} \right) \text{ OR } 212,500 \left[ 1 - \left( \frac{(1.00532)^{204} - 1}{(1.00532)^{360} - 1} \right) \right] = 140,418.73

Loan B: $1,402.94 \left( \frac{1 - \left( \frac{1}{1.00577} \right)^{156}}{.00577} \right) \text{ OR } 212,500 \left[ 1 - \left( \frac{(1.00577)^{204} - 1}{(1.00577)^{360} - 1} \right) \right] = 144,044.58

Finally, twenty-eight years into the 30-year payment period (with 336 months gone and 24 remaining) the amounts owed are:

Loan A: $1,326.97 \left( \frac{1 - \left( \frac{1}{1.00532} \right)^{24}}{.00532} \right) \text{ OR } 212,500 \left[ 1 - \left( \frac{(1.00532)^{336} - 1}{(1.00532)^{360} - 1} \right) \right] = 29,823.78

Loan B: $1,402.94 \left( \frac{1 - \left( \frac{1}{1.00577} \right)^{24}}{.00577} \right) \text{ OR } 212,500 \left[ 1 - \left( \frac{(1.00577)^{336} - 1}{(1.00577)^{360} - 1} \right) \right] = 31,358.89

Note the outcome: at any time during the 30-year amortization period (other than the end of month 0, when $212,500 is owed on either loan, and the end of month 360, when $0 is owed on either loan), less principal remains to be paid on the loan with the same original principal and the lower interest rate. A loan with a higher interest rate amortizes less rapidly — even though the monthly payments are higher in keeping with the higher interest rate. This peculiar outcome appears to happen for all interest rate possibilities; I can not find an exception.
The easiest way to compute the desired values with a financial calculator is to start with the first loan: enter $212,500 +/- PV, $0 FV, 360 N, 6.384 ÷ 12 = I/Y; CPT PMT, it should show $1,326.97. Then enter 6 x 12 = N, CPT FV; it should show $195,317.76. Then enter 17 x 12 = N, CPT FV; it should show $140,418.73. Then enter 28 x 12 = N, CPT FV; it should show $29,823.78. Then for the second loan: enter $0 FV, 360 N, 6.924 ÷ 12 = I/Y (the -$212,500 remains intact); CPT PMT, it should show $1,402.94. Then enter 6 x 12 = N, CPT FV; it should show $196,772.84. Then enter 17 x 12 = N, CPT FV; it should show $144,044.58. Enter 28 x 12 = N, CPT FV; should show $31,358.89.

A more detailed financial calculator approach would be to use the amortization keys. Once the $1,326.97 payment has been computed for the first loan type 2nd AMORT 1 ENTER ↓ 72 ENTER ↓ (it should show $195,317.76 as the principal still owed 72 months into the loan’s life); then ↑ 204 ENTER ↓ (it should show $140,418.73); then ↑ 336 ENTER ↓ (it should show $29,823.78). And once the $1,402.94 payment has been computed for the second loan type 2nd AMORT 1 ENTER ↓ 72 ENTER ↓ (it should show $196,772.84); then ↑ 204 ENTER ↓ (it should show $144,044.58); then ↑ 336 ENTER ↓ (it should show $31,358.89).

10. Compute the lender’s yield to maturity, at the origination date, for each of the following mortgage loans.

<table>
<thead>
<tr>
<th>Loan</th>
<th>Monthly Payment</th>
<th>Maturity in Months</th>
<th>Amount Borrowed</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$1,020</td>
<td>360</td>
<td>$160,000</td>
</tr>
<tr>
<td>B</td>
<td>$1,145</td>
<td>300</td>
<td>$175,000</td>
</tr>
<tr>
<td>C</td>
<td>$1,680</td>
<td>180</td>
<td>$205,000</td>
</tr>
</tbody>
</table>

“Yield to maturity” is bond market terminology for a money lender’s effective annual rate (EAR) of return. Here no points are charged and no prepayments are anticipated, so the procedure for computing the YTM (EAR) is very straightforward. For Loan A, the borrower leaves the bank today with $160,000 and pays $1,120 back 360 times. We compute the lender’s periodic percentage return using trial and error to solve for r [and then take (1 + r)^12 – 1 ] in the following equation:

\[
1,020.00 \left( \frac{1- \frac{1}{(1+r)^{360}}}{r} \right) = 160,000.
\]

With a financial calculator, enter $160,000 +/- PV, $0 FV, 360 N, $1,020 PMT, CPT I/Y. The calculator handles the trial and error computations quickly for you; it should show .548537. Since that is a monthly periodic rate, APR is .548537% x 12 = 6.582449% and EAR is \((1.00548537)^{12} - 1 = 6.7847\%\). Because the $1,020 payments are made monthly and the 360 periods are months, the rate that the calculator computes is a monthly rate. But we talk about rates of return in annual terms, to provide some degree of comparability across a range of investment situations. Also remember to enter either the PV or the PMT as a negative amount (and the other one as positive); otherwise the calculator can not solve for I/Y (it would be as though a lender received money up front and then received a series of payments; the rate of return would be infinite - a concept the calculator can not work with).

For Loan B, the borrower leaves the bank today with $175,000 and pays $1,145 back 300 times. We compute the lender’s percentage return using trial and error to solve for r [and then take (1 + r)^12 – 1 ] in the following equation:
\[
\$1,145.00 \left( \frac{1 - \left( \frac{1}{1+r} \right)^{300}}{r} \right) = \$175,000
\]

With a financial calculator (we will assume that the FV register already holds $0 from our Loan A entries), enter $175,000 +/- PV, 300 N, $1,145 PMT, CPT I/Y. The calculator should show .513561% x 12 = 6.27371% and an EAR of \((1.00513561)^{12} - 1 = 6.3398\%\).

For Loan C, the borrower leaves the bank today with $205,000 and pays $1,680 back 180 times. We compute the lender's percentage return using trial and error to solve for \(r\) [and then take \((1 + r)^{12} - 1\)] in the following equation:

\[
\$1,680.00 \left( \frac{1 - \left( \frac{1}{1+r} \right)^{300}}{r} \right) = \$205,000
\]

With a financial calculator (again we assume that the FV register already holds $0 from our Loan A entries), enter $205,000 +/- PV, 180 N, $1,680 PMT, CPT I/Y. The calculator should show .462144% x 12 = 5.545731% and an EAR of \((1.00462144)^{12} - 1 = 5.6889\%\).

11. How many discount points would a lender have to charge to provide a yield to maturity (an EAR measure) of 6.5% on a $215,000 loan with 30 years of $1,290 monthly payments? How many discount points would have to be charged so the lender would earn a 7.4% EAR on a $305,000 loan with 20 years of $2,365 monthly payments?

This one may seem scary because we are not used to using points in this context. But it's not that bad if we just think through the process with our now-familiar skeleton. Consider the first loan. If a bank lent $215,000 and the borrower paid back $1,290 each month for 30 years, the bank would obviously be earning a positive rate of return (since \(360 \times \$1,290 = \$464,400 > \$215,000\) invested). The solution is a trial and error IRR computation; the monthly periodic rate turns out to be .500583%; double check:

\[
\$1,290 \left( \frac{1 - \left( \frac{1}{1.00500583} \right)^{360}}{.00500583} \right) = \\
\$1,290 \times 166.6666667 = \$215,000. \checkmark
\]

So with a monthly periodic \(r\) of .00500583 the accompanying APR would be .00500583 x 12 = 6.006996% and the EAR would be \((1.00500583)^{12} - 1 = 6.175172\%\). So far, so good. But the bank has said it wants the $1,290 payment stream to provide a 6.5% EAR; how can they make it happen?

One way to increase the lender's effective return is to have to borrower pay more each month, but we are told that the payment will be $1,290 each month. Another way is to force the borrower to pay something "up front," in the form of discount points. In effect, the lender just hands the borrower fewer net dollars at closing, after computing payments based on a higher loan amount. (The money provider's rate of return relates what she gets back to what she gives up; we can increase a lender's return - or borrower's cost - by having the lender hand over fewer dollars in return for an identified payment stream, or having the borrower pay more, periodically or in a lump sum, to get the identified loan proceeds.) Let's word the problem differently: A bank lends $215,000 at a stated, nominal interest rate of 6.006996%, which corresponds to a monthly periodic
rate of .06006996 ÷ 12 = .00500583, or an EAR of \((1.00500583)^{12} - 1 = 6.175172\%\). But the bank wants to earn an effective return of 6.5\%. How many discount points should the lender charge (if it is able to do so) to increase its investment yield from 6.175172\% to 6.5\%?

Let’s begin by computing the monthly payment, as we have done so many times before:

\[
PMT \left(1 - \frac{1}{1.00500583^{360}} \right) = $215,000
\]

\[
PMT \times 166.666596 = $215,000
\]

So \(PMT = $215,000 \div 166.666596 = OR PMT = $215,000 \times .006000004 = $1,290.00\).

We knew this already, of course, but it can be helpful to retrace the steps we understand before moving to new ground. Note that 166.666596 is the present value of an annuity factor for a loan with a 6.175172\% EAR. Let’s find the present value of an annuity factor for a loan with a 6.5\% EAR, or a monthly periodic rate of \(\sqrt[12]{1.065} - 1 = (1.065^{1/12} - 1) = (1.065^{0.083333} - 1) = .5261694\) or .005261694:

\[
\left(1 - \frac{1}{1.005261694^{360}} \right) = 161.3195083.
\]

Now let’s compute the present value of a stream of 360 payments of $1,290 each at a .5261694\% monthly periodic rate:

\[
$1,290 \left(1 - \frac{1}{1.005261694^{360}} \right) = TOT
\]

\[
$1,290 \times 161.319514 = $208,102.17.
\]

Finally, compute $215,000 (the basis for computing loan payments) minus $208,102.17 (the net amount the bank actually wants to hand the borrower) = $6,897.83 (the amount the bank should ask the borrower to pay up-front, in the form of discount points). Because a point would be 1\% of the stated loan amount, or $2,150, the bank here should charge the borrower about $6,897.83 ÷ $2,150 = 3.20829 points.

To solve with your financial calculator, enter \(0 FV, 360 N, 1,290 PMT, 1.065 y^x \left(1 \div 12\right) = -1 \times 100 = I/Y \) (to compute a monthly periodic rate of .526169\% from a 6.5\% EAR – typically the annual rate we are given is an APR and we just divide by 12); CPT PV. It should show -$208,102.17. Because this amount is negative, simply add the $215,000 stated loan amount to get the dollar value of points: $6,897.83. To get the appropriate percentage figure (the number of points), enter ÷ $215,000 = x 100 = ; it should show 3.20829.

Now that we understand the process, we can approach the second loan by fast-forwarding to the step where we multiply the monthly payment by the appropriate present value of an annuity factor. Let’s find the PV of an annuity factor for a 240-month loan with a 7.4\% EAR, or a monthly periodic rate of \(\sqrt[12]{1.074} - 1 = (1.074^{1/12} - 1) = (1.074^{0.083333} - 1) = .5261694\) or .005966898:
\[
\left( 1 - \left( \frac{1}{1.005966898} \right)^{240} \right) = 127.3968531.
\]

Now let's compute the present value of a stream of 240 payments of $2,365 each at a .5966898% monthly periodic rate:

\[
$2,365 \left( 1 - \left( \frac{1}{1.005966898} \right)^{360} \right) = \text{TOT}
\]

\[
$2,365 \times 127.3968531 = $301,293.56.
\]

Finally, compute $305,000 (the basis for computing loan payments) minus $301,293.56 (the net amount the bank actually wants to hand the borrower) = $3,706.44 (the amount the bank should ask the borrower to pay up-front, in the form of discount points). Because a point would be 1% of the stated loan amount, or $3,050, the bank here should charge the borrower about $3,706.44 ÷ $3,050 = 1.215227 points.

To solve with your financial calculator, enter $0 FV, 240 N, $2,365 PMT, 1.074 y x (1 ÷ 12) = -1 = x 100 = I/Y (to compute a monthly periodic rate of .596690% from a 7.4% EAR - typically the annual rate we are given is an APR and we just divide by 12); CPT PV. It should show -$301,293.56. Because this amount is negative, simply add the $305,000 stated loan amount to get the dollar value of points: $3,706.44. To get the appropriate percentage figure (the number of points), enter ÷ $305,000 = x 100 = ; it should show 1.215227.

12. Local lenders are offering the following terms on 30-year, fixed rate mortgage loans: a 6.39% stated annual interest rate with no discount points, and a 6.126% stated annual interest rate with 2 discount points. Assume that the amount nominally borrowed is $245,000. Which type loan is accompanied by the lowest cost of borrowing (APR and EAR) under each of the following scenarios?

- Scenario a: the borrower makes regular payments for the entire 30 years.
- Scenario b: the borrower makes regular payments for 7 years and then pays the remaining principal balance.
- Scenario c: the borrower makes regular payments for 7 years, and then pays the remaining principal balance even though the lender imposes a 3% prepayment penalty.

Here we have another trial and error exercise, for which we would use the IRR function on a spreadsheet or the I/Y or IRR key on a financial calculator. (Now we phrase things in terms of the borrower's cost, but the borrower's cost and the lender's return are identical unless the two are brought together by a middle party that receives a lump-sum or periodic payment, such that the lender does not receive everything that the borrower pays. In this situation there is no identified middle party.) For the exam you should be able either to set the problem up, or to solve it with a financial calculator. Let's deal with each loan under all three scenarios.

**FIRST LOAN**: It carries no discount points, such that the borrower receives, at the loan’s closing, all of the money nominally borrowed. So the borrower leaves the lender’s office with $245,000 more than she entered with. With a monthly periodic interest rate of 6.39% ÷ 12 = .5325% and 30 x 12 = 360 monthly payment periods, we compute:
\[
PMT \times FAC = TOT
\]
\[
PMT \left( 1 - \left( \frac{1}{1.005325} \right)^{360} \right) = 245,000
\]
\[
PMT \times 160.028079 = 245,000
\]
So \( \text{PMT} = \frac{245,000}{160.038079} \) OR \( \text{PMT} = 245,000 \times 0.006249 = 1,530.89 \).

If there is no prepayment, then the borrower leaves the bank today with $245,000 and pays $1,530.89 back 360 times. Because $1,530.89 \times 360 = 551,120.40$, an amount far more than the $245,000 net amount lent, the transaction clearly represents a positive periodic rate of return for the lender. How big a return? For Scenario a we must use trial and error to solve for \( r \) in the following equation (just like we computed in problem 9 above):

\[
1,530.89 \left( 1 - \frac{1}{1+r} \right)^{360} = 245,000.
\]

If the loan is prepaid after 7 years, the equation becomes a bit more complicated. First we must find the amount to be paid at the end of year 7, along with the 84\(^{th} \) $1,530.89 payment, which is

\[
1,530.89 \left( \frac{1}{1.005325} \right)^{276} \text{ OR } 245,000 \left[ 1 - \left( \frac{(1.005325)^{84} - 1}{(1.005325)^{360} - 1} \right) \right] = 221,110.77
\]

Then to find the lender's return for Scenario b, we use trial and error to solve for \( r \) in the following equation:

\[
1,530.89 \left( \frac{1}{1+r} \right)^{84} + 221,110.77 \left( \frac{1}{1+r} \right)^{84} = 245,000.00.
\]

And if the loan is prepaid after 7 years and there is a 3\% prepayment penalty, then the amount paid at the end of year 7, along with the 84\(^{th} \) $1,530.89 payment, is 1.03 \times 221,110.77 = 227,744.09.

We compute the lender's return in Scenario c by using trial and error to solve for \( r \) in the following equation:

\[
1,530.89 \left( \frac{1}{1+r} \right)^{84} + 227,744.09 \left( \frac{1}{1+r} \right)^{84} = 245,000.00.
\]

In each of the 3 scenario solution equations above, when we solve for \( r \) we have found the monthly periodic rate of return to the lender (which is also the monthly periodic cost to the borrower, since everything paid by the borrower is received by the lender - there is no middle party that brings the two together and takes an up-front fee or a portion of each payment). So on an exam question you might simply show the appropriate equation and then say, "Solve for \( r \) with trial and error to get the monthly rate; take \( r \times 12 \) to get the APR and \( (1 + r)^{12} - 1 \) to get the EAR."
If you use a financial calculator you can let it do the trial and error computations (and the other needed computations) for you. Compute the loan payment by hitting $245,000 +/- PV, $0 FV, 360 N, 6.39 ÷ 12 = I/Y (the periodic interest rate is 6.39% ÷ 12 = .5325%, or .005325); CPT PMT. It should show $1,530.89, as we computed manually above. Then for Loan 1, to solve for Scenario a, just hit CPT I/Y; it should show .5325. (If that seems like circular logic, in a sense it is. If the lender hands over $245,000 and gets 360 monthly payments of $1,530 that were based on $245,000 in principal and a .5325% monthly rate, then the lender’s return is .5325% per month.) Since that is a monthly periodic rate, APR is .5325% x 12 = 6.39% and EAR is \((1.00^{.5325})^{12} - 1 = 6.5805\%\).

Then for Scenario b, you have to find the balance owed after 84 months. If the values for scenario a are still entered you can just enter 84 N, CPT FV; it should show $221,110.77. Then enter CPT I/Y; it should show .5325. So once again the APR is .5325% x 12 = 6.39% and the EAR is \((1.005325)^{12} - 1 = 6.5805\%\). Here the lender receives .5325% per month on whatever amount of principal remains outstanding, for as long as it is outstanding. Prepayment does not increase the periodic cost/APR/EAR on a loan if there are no points or prepayment penalties.

Finally for Scenario c, you have to adjust the balance owed after 60 months for the 3% prepayment penalty. Hit recall FV; it should show $221,110.77. Hit x 1.03 = FV (it should show $227,744.09), CPT I/Y; it should show .558922. So the APR is .558922% x 12 = 6.7071% and the EAR is \((1.00558922)^{12} - 1 = 6.9171\%\).

SECOND LOAN: It carries 2 discount points, such that the borrower must pay the lender 2.00% of the $245,000 stated principal as additional up-front fees (sometimes thought of as interest). Thus the borrower leaves the lender’s office with only 98.00% of $245,000, or $240,100 net of what she had when she entered. But the payment is computed based on the entire $245,000 stated principal. With a monthly periodic interest rate of 6.126% ÷ 12 = .5105% and 360 monthly payment periods, we compute:

\[
PMT \left(1 - \frac{1}{(1.005105)^{360}}\right) = $245,000
\]

\[
PMT \times 164.561592 = $245,000
\]

So \(PMT = \$245,000 \div 164.561592\) OR \(PMT = \$245,000 \times .006077 = $1,488.80\).

If there is no prepayment, then the borrower leaves the bank today with $240,100 and pays $1,488.80 back 360 times. What percentage return does that represent for the lender? For Scenario a we must use trial and error to solve for r in the following equation:

\[
$1,488.80 \left(1 - \frac{1}{(1+r)^{360}}\right) = $240,100.
\]

Note that points do not reduce the loan payments; they merely reduce the net amount that the lender gives up to get the payment stream. (The lender only gives up $240,100 to get back
$245,000 plus .5105% per month in interest - so the rate of return will end up being more than .5105% per month.)

If the loan is prepaid after 7 years, then the amount paid at the end of year 7, along with the 84th $1,488.80 payment, is

$$1,488.80 \left( \frac{1-(1.005105)^{84}}{r} \right) + 220,106.81 \left( \frac{1}{1+r} \right)^{84} = 240,100.$$ 

To find the lender's return for Scenario b, we use trial and error to solve for r in the following equation:

$$1,488.80 \left( \frac{1-(1.005105)^{84}}{r} \right) + 220,106.81 \left( \frac{1}{1+r} \right)^{84} = 240,100.$$ 

And if the loan is prepaid after 7 years and there is a 3% prepayment penalty, then the amount paid at the end of year 7, along with the 84th $1,488.80 payment, is 1.03 x $220,106.81 = $226,710.02.

We compute the lender's return in Scenario c with trial and error, solving for r in the equation:

$$1,488.80 \left( \frac{1-(1.005105)^{84}}{r} \right) + 226,710.02 \left( \frac{1}{1+r} \right)^{84} = 240,100.$$ 

In each of the 3 scenario solution equations above, when we solve for r we have found the monthly periodic rate. So on an exam question you might simply show the appropriate equation and then say, "Solve for r with trial and error; take the monthly rate r x 12 to get the APR and (1 + r)^{12} - 1 to get the EAR."

With a financial calculator, compute the loan payment by hitting $245,000 +/- PV, $0 FV, 360 N, 6.126 ÷ 12 = I/Y (the periodic interest rate is 6.126% ÷ 12 = .5105%, or .005105); CPT PMT. It should show $1,488.80, as above. Then for Loan 2, to solve for Scenario a, enter .98 x $245,000 +/- = PV (replacing the original -$245,000 PV) and CPT I/Y; it should show .52641. Since that is a monthly periodic rate, APR is .526411% x 12 = 6.316929% and EAR is (1.00526411)^{12} - 1 = 6.5031%.

Then for Scenario b you have to find the balance owed after 84 months. Re-enter $245,000 +/- PV, $0 FV, 360 N, 6.126 ÷ 12 = I/Y; CPT PMT and it should show $1,488.80. Then enter 84 N, CPT FV; it should show $220,106.81. Then enter .98 x $245,000 +/- = PV (it should show -$240,100), CPT I/Y; it should show .541525. So the APR is .541525% x 12 = 6.498299% and the EAR is (1.00541525)^{12} - 1 = 6.6954%.

Finally for Scenario c, you have to adjust the balance owed after 84 months for the 3% prepayment penalty. Hit recall FV; it should show $220,106.81. Hit x 1.03 = FV (it should show $226,710.02), CPT I/Y; it should show .568087. So the APR is .568087% x 12 = 6.817038% and the EAR is (1.00568087)^{12} - 1 = 7.0341%.

So now, after all that computing, which loan represents the best deal for a borrower? Let's summarize the effective annual rate (EAR) of return figures here:
Notice the outcome. If the borrower plans to keep the loan intact until maturity Loan 2, with the lower stated interest rate, has the lower cost, despite the required payment of points, because the benefit of the lower payments is received for the entire 360 months (and the fixed cost of the points is spread over the entire 360 months). But if repayment is to occur after 7 years the loan with the higher stated rate but with no points has the lower cost – whether there is a prepayment penalty or not. In choosing from among loan alternatives you have to look at both the stated interest rate and other costs, such as points. It is generally the case that the borrower does not reduce the loan’s periodic cost by paying points to “buy down” the interest rate if the loan will be prepaid at an early date, although we have to compute based on the specific numbers in a given situation to know for sure.

13. An individual borrows $250,000, with repayment to be based on a contract interest rate (APR) of 5.55%. But the lender allows the borrower to choose among various payment levels. How many months would it take to retire the loan (pay back all principal and applicable interest) if the amount he pays at the end of each month is $1,832.10? What is he pays only $1,427.33 per month? What is he pays an even smaller $1,156.25 each month – or less?

Here we are just trying to solve for the number of time periods in our now-familiar loan repayment structure. With $250,000 in principal, a 5.55% ÷ 12 = .4625% monthly periodic interest rate, and an $1,832.10 monthly payment, we use logarithms to solve for n in

\[ \text{PMT} \times \text{Fac} = \text{Tot} \]

\[ $1,832.10 \left( \frac{1 - \left( \frac{1}{1.004625} \right)^n}{.004625} \right) = $250,000 \]

\[ \left( \frac{1 - \left( \frac{1}{1.004625} \right)^n}{.004625} \right) = 136.455434 \]

\[ 1 - \left( \frac{1}{1.004625} \right)^n = .631106 \]

\[ \left( \frac{1}{1.004625} \right)^n = .368894 \]

\[ (.995396)^n = .368994 \]

\[ n \ln (.995396) = \ln .368994 \]

\[ n (-.004614) = .997247 \]

\[ n = 216.119207, \text{ just over 216 months.} \]
Financial calculator solution: enter $250,000 +/- PV, $0 FV, $1,832.10 PMT, 5.55 \div 12 = I/Y; CPT N. The calculator should show 216.119207. Here the borrower is paying more each month than would be required for repaying the loan over a standard 360 month amortization period.

With a $1,427.33 monthly payment, we solve for n in

$$1,427.33 \left( 1 - \frac{1}{1.004625} \right)^n = 250,000$$

$$\left( 1 - \frac{1}{1.004625} \right)^n = 175.152207$$

$$1 - \left( \frac{1}{1.004625} \right)^n = .810079$$

$$\left( \frac{1}{1.004625} \right)^n = .189921$$

$$(.995396)^n = .189921$$

$$n \ln (.995396) = \ln .368994$$

$$n (-.004614) = -1.661147$$

$$n = 359.996824$$, just about exactly 360 months.

Financial calculator solution: enter $250,000 +/- PV, $0 FV, $1,427.33 PMT, 5.55 \div 12 = I/Y; CPT N. The calculator should show 359.996824. $1,427.33 is the payment that would cause the loan to amortize over the standard 360 month period. (Our computed answer differs a tiny bit from exactly 360 because the payment that truly amortizes the loan over exactly 360 months is $1,427.325097, whereas we used a rounded figure of $1,427.33.)

Finally, with an $1,156.25 monthly payment, we solve for n in

$$1,156.25 \left( 1 - \frac{1}{1.004625} \right)^n = 250,000$$

$$\left( 1 - \frac{1}{1.004625} \right)^n = 216.216216$$

$$1 - \left( \frac{1}{1.004625} \right)^n = 1.000000$$

$$\left( \frac{1}{1.004625} \right)^n = 0.000000$$
\[(.995396)^n = 0.000000\]
\[n \ln (0.995396) = \ln 0.000000\]
\[n (-.004614) = \text{ERROR}\]
\[n = \text{Can not be solved}\]

Financial calculator solution: enter $250,000 +/- PV, $0 FV, $1,156.25 PMT, 5.55 \div 12 = \text{I/Y}; \text{CPT N}. The calculator shows an Error message. Why? Mechanically speaking, 0 has no logarithm. (There is no power to which we could raise the base of natural logarithms \(e = 2.7182818\) … to get 0. The ln of 1 is 0, because \(e^0 = 1\), but the ln of 0 does not exist.) Conceptually speaking, the problem is that this loan would require an infinite time horizon for repayment - note that the $1,156.25 monthly payment divided by the $250,000 original principal is $1,156.25 \div $250,000 = .004625, which is the monthly interest rate. Or turn it around and multiply that monthly rate by the original principal: .004625 \times $250,000 = $1,156.25. If someone borrows $250,000 at a 5.55% APR, then a payment of $1,156.25 is just enough to cover the interest each month, leaving the $250,000 in principal fully intact. This loan would ultimately require a full balloon payment of $250,000; the monthly payments are just enough to cover interest and pay nothing toward principal.

If you tried a payment less than $1,156.25 you would also get an error message of course; the monthly payment would not even be high enough to cover the interest in month 1, so there would be negative amortization - the amount owed would grow increasingly above $250,000 month after month. There have been real-world examples of negative amortization mortgage loans in the past, including graduated payment loans and the option adjustable rate mortgage, or Option ARM, loans that were roundly criticized during the mortgage meltdown and financial market crisis. But these types of loans may be history; rules put in place by the Consumer Financial Protection Bureau in 2013 would prevent any loan with negative amortization from receiving the Qualified Mortgage status that protects lenders from being sued by borrowers who encounter repayment difficulties.

14. You borrow $214,000 through a mortgage loan with a 5.04% APR interest rate and a 30-year amortization period. The required monthly payment is $1,154.04, but for 11 years you make a somewhat higher payment of $1,250 per month. If you then increase your end-of-month payment to $1,750, how long will it take for all principal to be repaid?

Here we are 11 years or 132 months into the amortization period, and the periodic interest rate is .0504 \div 12 = .0042. The easiest way to set the problem up is to solve for \(n\) in the equation

\[
$1,250 \left( \frac{1 - \left( \frac{1}{1.0042} \right)^{132}}{.0042} \right) + $1,750 \left( \frac{1 - \left( \frac{1}{1.0042} \right)^n}{.0042} \right) \left( \frac{1}{1.0042} \right)^{132} = $214,000
\]

\[
$1,250 (101.170455) + $1,750 \left( \frac{1 - \left( \frac{1}{1.0042} \right)^n}{.0042} \right)(.575084) = $214,000
\]

\[
$126,463.07 + $1,006.40 \left( \frac{1 - \left( \frac{1}{1.0042} \right)^n}{.0042} \right) = $214,000
\]
$1,006.40 \left( \frac{1 - \left( \frac{1}{1.0042} \right)^n}{0.0042} \right) = 87,536.94

\left( \frac{1 - \left( \frac{1}{1.0042} \right)^n}{0.0042} \right) = 86.980512

1 - \left( \frac{1}{1.0042} \right)^n = .365318

- \left( \frac{1}{1.0042} \right)^n = -.634682 \quad \text{SO} \quad \left( \frac{1}{1.0042} \right)^n = .634682 \quad \text{OR} \quad (.995818)^n = .634682

n \ln (.995818) = \ln .634682

n (.004191) = -.454631

n = 108.472722, \text{ or about 108} \frac{1}{2} \text{ months.}

All principal would be repaid after 132 + 108.5 = 240.5 months or just over 20 years; by paying the higher amounts described each month the borrower would have the loan repaid about 10 years early.

In a more complicated approach, but one that is friendlier for financial calculator use, we would start by finding the amount still owed at the end of year 11/month 132 (with 360 - 132 = 228 months remaining) under the 30-year amortization:

$1,154.04 \times \left( \frac{1 - \left( \frac{1}{1.0042} \right)^{228}}{0.0042} \right) \quad \text{OR} \quad 214,000 \left[ 1 - \left( \frac{(1.0042)^{132} - 1}{(1.0042)^{360} - 1} \right) \right] = 169,098.24

(such that principal already repaid is $214,000 - $169,098.24 = $44,901.76). Then think as though you have made a savings deposit of another $1,250 - $1,154.04 = $95.96 each month. If invested at the mortgage lending rate, then the savings fund, which could be used to prepay some principal (it's actually done systematically within the loan) would have grown over 132 months to

$95.96 \times \left( \frac{(1.0042)^{132} - 1}{0.0042} \right) = 95.96 \times (175.922891) = 16,882.36

So what has been repaid over 132 months is $44,901.76 + $16,882.36 = $61,784.12, such that principal still owed is $214,000 - $61,784.12 = $152,215.88 (or just $169,098.24 - $16,882.36 = $152,215.88). Then compute the number of months it takes for $152,215.88 to be retired with $1,750 monthly payments:

$1,750 \left( \frac{1 - \left( \frac{1}{1.0042} \right)^n}{0.0042} \right) = 152,215.88

Solve for n with logarithms: the solution turns out to be 108.472722, which added to the 132 months already passed totals about 240.5 months, as shown above. Financial calculator: $214,000 +/- PV, $0 FV, 360 N, 5.04 ÷ 12 = I/Y; CPT PMT. Shows $1,154.05. Enter 228 N, CPT PV: shows -
$169,098.24. Enter +/- STO 1 to store that value in memory register 1. Then enter $1,250 - RCL PMT =; shows $95.96. Hit the PMT key to set a new payment of $95.96, then 132 N, $0 PV, CPT FV; shows -$16,882.36. Enter + RCL 1 =; it shows $152,215.88. Enter +/- PV, $0 FV, $1,750 PMT, CPT N; shows 108.472722. Finally enter + 132 = to get the 240.472722 months combined answer.

15. $199,000 is borrowed through a fixed-rate, fixed-payment mortgage loan with an interest rate that is quoted as a 6.072745% EAR. End-of-month payments are to be made over 25 years. Two discount points are charged by the lender. Calculate the effective annual cost of borrowing (in APR and EAR terms) if payments continue for the entire 25-year amortization period, and if the remaining principal balance is prepaid after 16 years and 9 months?

First we must compute the monthly payment; With an annual interest rate given as a 6.072745% EAR and monthly payments, our monthly periodic interest rate is $\sqrt{1.06072745} - 1 = .004925$, or .4925%. With 25 years of monthly payments, we have $25 \times 12 = 300$ payment periods. (Note that the accompanying APR would be $.4925 \times 12 = 5.91\%$. Therefore, with $199,000 borrowed we compute

$$PM(T \left(1 - \frac{1}{1 + .004925}\right)^{300}\right) = 199,000$$

$$PM \times 156,540,796 = 199,000$$

So $PM = 199,000 / 156,540,796$ OR $PM = 199,000 \times .006388 = 1,271.23$.

Financial calculator: $199,000 +/- PV, $0 FV, 25 \times 12 = N, .4925 = I/Y$ (or say 5.91 \div 12 = I/Y); CPT PMT. Shows $1,271.23.

Recall that two points were charged, such that the borrower agreed to pay back $198,000 plus .4925% per month interest, but got the use of only $198,000 = 195,020. So if all payments are to be made as scheduled over the 25 years we solve for $r$, with trial and error, in the equation

$$1,271.23 \left(1 - \frac{1}{1 + r}\right)^{300} = 195,020$$

(Recall that discount points do not affect the payment; they affect only what the borrower receives at closing in return for the promise to make the payments.) It turns out to be $.510260\%$ (do you see why the answer would be simply .4925%, with a 5.91% APR, if no points were charged?). The accompanying APR $0.00510260 \times 12 = 6.123115\%$; EAR is $(1.00510260)^{12} - 1 = 6.2979\%$. Financial calculator: with all of the values from above still intact enter $0.98 \times 199,000 = PV; CPT I/Y$. It goes blank very briefly while doing the trial and error calculations, then shows .510260.

What if prepayment is to occur after 16 years and 9 months? The partial year causes no special concerns; this problem has us in a world of months, where yearly measures serve only as a baseline for computing monthly measures. So the time measure of interest is $(16 \times 12) + 9 = 201$ months. Principal that remains unpaid 201 months into a 300-month repayment period (99 months remaining) is
\[
1,271.23 \left( 1 - \left( \frac{1}{1.004925} \right)^{99} \right) \text{ OR } 199,000 \left[ 1 - \left( \frac{(1.004925)^{201} - 1}{(1.004925)^{300} - 1} \right) \right] = 99,414.59.
\]

So now we solve for \( r \), with trial and error, in the equation
\[
1,271.23 \left( 1 - \left( \frac{1}{1+r} \right)^{201} \right) + 99,414.59 \left( \frac{1}{1+r} \right)^{201} = 195,020.
\]

The answer turns out to be \( .511670 \) per month, for an APR of \( .00511670 \times 12 = 6.140038\% \) and an EAR of \((1.00511670)^{12} - 1 = 6.3158\% \). Financial calculator: \( 199,000 +/- \text{ PV}, 0 \text{ FV}, 25 \times 12 = \text{ N}, .4925 \text{ I/Y} \) (or say 5.91 \div 12 = \text{ I/Y}); \text{ CPT PMT}. Shows \( 1,271.23 \). Then (since 201 months will have passed) type 201 N; \text{ CPT FV}. Should show \( 99,414.59 \). Type .98 x $199,000 = and the \( \pm \) key (or just \( -$195,020 \)) and then \( \text{ PV}; \text{ then CPT I/Y}. \) Should show \( .511670 \).

16. You borrow \$224,000 through a fixed-rate mortgage loan with end-of-month payments for 30 years. The stated interest rate is a 6.363% APR. This loan has an 8/22 interest only provision, meaning that the first 8 years’ payments are limited to interest owed on the remaining principal balance; full amortization then occurs over the loan’s remaining 22-year life.

a. Calculate the monthly payment for years 1 – 8, and then for years 9 – 30.

There is no amortization (positive or negative) during the first 8 years, so the amount of principal owed remains intact and each month’s payment is simply the \( .06363 \div 12 = .0053025 \) monthly interest rate, multiplied by the \$224,000 borrowed: \( .0053025 \times \$224,000 = \$1,187.76 \).

Then because no principal has been repaid over the first 8 years, we essentially start out in year 9 with a new \$224,000, 22-year (264 month) loan with a 6.363% stated annual (53025% monthly) interest rate:

\[
\text{PMT} \left( 1 - \left( \frac{1}{1.0053025} \right)^{264} \right) = 224,000
\]

\[
\text{PMT} \times 141.905649 = 224,000
\]

So \( \text{PMT} = 224,000 \div 141.905649 \) OR \( \text{PMT} = 224,000 \times .007047 = \$1,578.51 \).

The payment has to be higher than the payment during the interest-only period, because now we are accounting for both regular interest payments (as above) plus a systematic retirement of principal. Financial calculator solution: \( 224,000 +/- \text{ PV}, 0 \text{ FV}, 22 \times 12 = \text{ N}, 6.363 \div 12 = \text{ I/Y}; \text{ CPT PMT}. \) Shows \$1,578.51.

b. Calculate the effective annual cost of borrowing (in APR and EAR terms) if payments continue for the entire 30 years.

The lender was entitled to collect \$224,000 plus interest, which in this case played out as 96 end-of-month payments of \$1,187.76 each, and then 264 end-of-month payments of \$1,578.51 each. So
we solve for \( r \) in

\[
$1,187.76 \left( \frac{1-(1+r)^{-96}}{r} \right) + $1,578.51 \left( \frac{1-(1+r)^{-264}}{r} \right) \left( \frac{1}{1+r} \right)^{96} = $224,000.
\]

Trial and error ultimately gives an answer of 0.530249\%, for an APR of 0.00530249 \times 12 = 6.362988\% and EAR of \((1.00530249)^{12} - 1\) = 6.5519\%. Double-check:

\[
$1,187.76 \left( \frac{1-(1.00530249)^{-96}}{0.00530249} \right) + $1,578.51 \left( \frac{1-(1.00530249)^{-116}}{0.00530249} \right) \left( \frac{1}{1+0.00530249} \right)^{96} = $224,000 \checkmark
\]

Financial calculator: with more than one multi-period payment stream we can not solve with the calculator’s simple time value function keys; have to use the IRR function. Type in CF \( 2^{nd} \) CLR WORK, $224,000 ± ENTER \downarrow, $1,187.76 ENTER \downarrow 96 ENTER \downarrow, $1,578.51 ENTER \downarrow 264 ENTER \downarrow, then hit IRR and CPT. The screen goes blank during the calculator’s trial and error computations; then should show 0.530249.

c. Calculate the effective annual cost of borrowing (in APR and EAR terms) if the remaining principal balance is repaid 17 years and 9 months into the loan’s original 30-year life.

Recall that because no principal is repaid during the 8 interest-only years, our amortization analysis applies only to the final 22 years (264 months) of the loan’s life. So 9 years and 9 months (117 months) into that 264 month amortization period (with 264 - 117 = 147 months remaining), the amount of principal that remains unpaid is

\[
$1,578.51 \left( \frac{1-(1.0053025)^{-147}}{0.0053025} \right) \text{ OR } $224,000 \left[ 1 - \left( \frac{(1.0053025)^{117}}{(1.0053025)^{264}} \right) \right] = $160,874.24
\]

Here the lender gives up $224,000 and then gets $1,187.76 at the end of each of months 1 - 96, $1,578.51 at the end of each of months 97 - 212 (116 months), and $160,874.24 along with the last $1,578.51 payment at the end of month 213 (for $1,578.51 + $160,874.24 = $162,452.75 total). Solve for \( r \) in

\[
$1,187.76 \left( \frac{1-(1+r)^{-96}}{r} \right) + $1,578.51 \left( \frac{1-(1+r)^{-117}}{r} \right) \left( \frac{1}{1+r} \right)^{96} + $160,874.24 \left( \frac{1}{1+r} \right)^{213} = $224,000
\]

OR

\[
$1,187.76 \left( \frac{1-(1+r)^{-96}}{r} \right) + $1,578.51 \left( \frac{1-(1+r)^{-116}}{r} \right) \left( \frac{1}{1+r} \right)^{96} + $162,452.75 \left( \frac{1}{1+r} \right)^{213} = $224,000
\]

Trial and error ultimately gives an answer of 0.530249\%, for an APR of 0.00530249 \times 12 = 6.362988\% and EAR of \((1.00530249)^{12} - 1\) = 6.5519\% -- the same as the answer to part b above!!
Why? Prepayment of principal does not, by itself, change the cost of borrowing. Here the borrower merely pays .530249% per month on any principal owed for as long as principal is owed. Double-check:

\[
\begin{align*}
$1,187.76 \left( \frac{1-(\frac{1}{1.00530249})}{.00530249} \right)^{96} + $1,578.51 \left( \frac{1-(\frac{1}{1.00530249})}{.00530249} \right)^{117} &= $160,874.24 \left( \frac{1}{1.00530249} \right)^{213} = $224,000 \\
\end{align*}
\]

Financial calculator: again we must use the IRR function. Type CF 2nd CLR WORK, $.98 x 224,000 = +/- ENTER ↓, $1,187.76 ENTER ↓ 96 ENTER ↓, $1,578.51 ENTER ↓ 116 ENTER ↓, $162,452.75 ENTER ↓ ↓, then hit IRR and CPT. The screen goes blank during the trial and error computations; then should show .530249.

d. Calculate the effective annual cost of borrowing (in APR and EAR terms) if the remaining principal balance is repaid 17 years and 9 months into the loan’s original 30-year life AND the lender charged two points at the closing.

Just as in part c above, principal that remains unpaid 117 months into the 264 month amortization period (with 264 - 117 = 147 months remaining), the amount of principal that remains unpaid is $160,874.24. As explained in part c, the lender gets $1,187.76 at the end of each of months 1 - 96, $1,578.51 at the end of each of months 97 - 212 (116 months), and $160,874.24 along with the last $1,578.51 payment at the end of month 213 (for $1,578.51 + $160,874.24 = $162,452.75 total). The only difference relative to part c is that here the lender gives up only 98% x $224,000 = $219,520 to get that stream of payments. So we solve for r in

\[
\begin{align*}
$1,187.76 \left( \frac{1-(\frac{1}{1+r})}{r} \right)^{96} + $1,578.51 \left( \frac{1-(\frac{1}{1+r})}{r} \right)^{117} \left( \frac{1}{1+r} \right)^{96} + $160,874.24 \left( \frac{1}{1+r} \right)^{213} &= $219,520 \\
\end{align*}
\]

OR

\[
\begin{align*}
$1,187.76 \left( \frac{1-(\frac{1}{1+r})}{r} \right)^{96} + $1,578.51 \left( \frac{1-(\frac{1}{1+r})}{r} \right)^{116} \left( \frac{1}{1+r} \right)^{96} + $162,452.75 \left( \frac{1}{1+r} \right)^{213} &= $219,520 \\
\end{align*}
\]

Trial and error ultimately gives an answer of .546920%, for APR of .00546920 x 12 = .6563036% and EAR of (1.00546920)\(^12\) - 1 = 6.7641%. Double-check:

\[
\begin{align*}
$1,187.76 \left( \frac{1-(\frac{1}{1.00546920})}{.00546920} \right)^{96} + $1,578.51 \left( \frac{1-(\frac{1}{1.00546920})}{.00546920} \right)^{117} \left( \frac{1}{1.00546920} \right)^{96} + $160,874.24 \left( \frac{1}{1.00546920} \right)^{213} &= $219,520 \\
\end{align*}
\]
Why is the answer not the same as the .530249% found in parts b and c above? First, because points are paid the borrower receives less (lender gives up less) for the same stream of payments shown in part c, the result of which is to increase the borrower's periodic cost (and lender's periodic return). Second, as we will discuss again with Chapter 8, whereas prepayment by itself does not change the periodic cost of borrowing, prepayment in the presence of points does.

Financial calculator: again we must use the IRR function. Type CF 2nd CLR WORK, $.98 x 224,000 = +/- ENTER ↓, $1,187.76 ENTER ↓ 96 ENTER ↓, $1,578.51 ENTER ↓ 116 ENTER ↓, $162,452.75 ENTER ↓ ↓, then hit IRR and CPT. The screen goes blank during the trial and error computations; then should show .546920.

17. You borrow $186,000 through a fixed-rate mortgage loan with a 5.46% APR initial stated annual interest rate and end-of-month payments to be made for 25 years. The lender charges three discount points, and there is a 7-year interest rate/reset, after which the rate can change annually with interest rate market conditions. (This type of loan, with a fixed interest rate period followed by a variable rate period, is sometimes called a “7/1 hybrid.”)

a. Calculate the monthly payment for years 1 – 7, and then calculate the monthly payment for year 8 if, at the end of year 7, the market interest rate on which payments are based has risen to a 6.15% APR.

A 5.46% stated annual interest rate and monthly payments gives us a .0546 ÷ 12 = .00455, or .455% monthly periodic rate. With $186,000 borrowed and a 25-year = 300 month loan life, we compute a year 1 – 7 monthly payment of

\[
PMT \left( 1 - \left( \frac{1}{1.00455^{300}} \right) \right) = 186,000
\]

\[
PMT \times 163.478559 = 186,000
\]

So \( PMT = \frac{186,000}{163.478559} \) OR \( PMT = 186,000 \times .006117 = 1,137.76 \).

Unpaid principal 7 years (84 months) into the 300-month amortization period (with 18 years = 216 months remaining), is

\[
1,137.76 \left( 1 - \left( \frac{1}{1.00455^{216}} \right) \right) OR 186,000 \left[ 1 - \left( \frac{1}{1.00455^{84}} \right) \right] = 156,262.02
\]

So now we think in terms of an 18 year (216 month) loan of $156,262.02 with a 6.15% stated annual interest rate (.0615 ÷ 12 = .5125% per month), and thus compute a monthly payment of

\[
PMT \left( 1 - \left( \frac{1}{1.005125^{216}} \right) \right) = 156,262.02
\]

\[
PMT \times 130.441967 = 156,262.02
\]

So \( PMT = \frac{156,262.02}{130.441967} \) OR \( PMT = 156,262.02 \times .007666 = 1,197.94 \).
Financial calculator: $186,000 +/- PV, $0 FV, 25 x 12 = N, 5.46 /12 = I/Y; CPT PMT. Shows $1,137.76. Then type 18 x 12 = N; CPT PV. Shows -$156,262.02. Type 6.15 /12 = I/Y; CPT PMT. Shows $1,197.94.

b. Calculate the effective annual cost of borrowing (in APR and EAR terms) if payments continue for the entire 30 years and the monthly payments can be expected to remain the same during all of years 8 – 30.

Recall that 3 discount points were charged, so the lender was entitled to collect $186,000 plus interest, but had to give up only .97 x $186,000 = $180,420 at the closing. Per the circumstances described here, the lender gives up a net $180,420 at closing, then collects 84 end-of-month payments of $1,137.76 each, and then anticipates getting 216 end-of-month payments of $1,197.94 each. So we solve for r in

$$1,137.76 \left( \frac{1 - (1 + r)^{-84}}{r} \right) + 1,197.94 \left( \frac{1 - (1 + r)^{-216}}{r} \right) \left( \frac{1}{1 + r} \right)^{84} = 180,420$$

Through trial and error we find r = .506653%, for an APR of .00506653 x 12 = 6.079840% and EAR of (1.00506653)^12 - 1 = 6.2522%. Double-check:

$$1,137.76 \left( \frac{1 - (1.00506653)^{-84}}{.00506653} \right) + 1,197.94 \left( \frac{1 - (1.00506653)^{-216}}{.00506653} \right) \left( \frac{1}{1.00506653} \right)^{84} = 180,420 \checkmark$$

Financial calculator: with more than one multi-period payment stream we must use the IRR function. Type CF 2nd CLR WORK, .97 x $186,000 = and the ± key (or just -$180,420) and ENTER ↓, $1,137.76 ENTER ↓ 84 ENTER ↓, $1,197.94 ENTER ↓ 216 ENTER ↓, then hit IRR and CPT. The screen goes blank during trial and error computations; then should show .506653.

c. Calculate the APR and EAR if payments remain the same each year after year 7 and the remaining principal balance is repaid at the end of year 19.

If remaining principal is repaid at the end of year 19 (12 years into the loan’s remaining 18 year period), then only 19 - 7 = 12 years (144 months) of payments at the higher 6.15% rate will have been paid by the borrower/received by the lender. Principal still unpaid 144 months into the 18-year or 216 month period (with 6 years = 72 months of payments remaining), if the interest rate is 6.15%/year = .5125%/month, and we have a $156,262.02 initial balance and $1,197.94 monthly payments, per part b above, is

$$1,197.94 \left( \frac{1 - (1.005125)^{-144}}{.005125} \right) OR 156,262.02 \left[ 1 - \left( \frac{1.005125^{144} - 1}{1.005125^{216} - 1} \right) \right] = 71,975.38$$

So now the lender gives up $180,420 and then gets $1,137.76 at the end of each of months 1 - 84, $1,197.94 at the end of each of months 85 - 228 (144 times), and $71,975.38 along with the last $1,197.94 payment at the end of month 228 (for $1,197.94 + $71,975.38 = $73,173.32 total). Solve for r in
$1,137.76 \left( \frac{1 - \left( \frac{1}{1+r} \right)^{84}}{r} \right) + $1,197.94 \left( \frac{1 - \left( \frac{1}{1+r} \right)^{144}}{r} \right) \left( \frac{1}{1+r} \right)^{84} + $71,975.38 \left( \frac{1}{1+r} \right)^{228} = $180,420

OR

$1,137.76 \left( \frac{1 - \left( \frac{1}{1+r} \right)^{84}}{r} \right) + $1,197.94 \left( \frac{1 - \left( \frac{1}{1+r} \right)^{143}}{r} \right) \left( \frac{1}{1+r} \right)^{84} + $73,173.32 \left( \frac{1}{1+r} \right)^{228} = $180,420

Trial and error ultimately gives an answer of .506426%, for an APR of .00506426 x 12 = 6.077111% and EAR of \((1.00506426)^{12} - 1\) = 6.2493%. Double-check:

$1,137.76 \left( \frac{1 - \left( \frac{1}{1.00506426} \right)^{84}}{.00506426} \right) + $1,197.94 \left( \frac{1 - \left( \frac{1}{1.00506426} \right)^{144}}{.00506426} \right) \left( \frac{1}{1.00506426} \right)^{84} + $71,975.38 \left( \frac{1}{1.00506426} \right)^{228} = $180,420

Financial calculator: again we must use the IRR function. Type CF 2nd CLR WORK,.97 x $186,000 = and the ± key (or just -$180,420) and ENTER ↓, $1,137.76 ENTER ↓ 84 ENTER ↓, $1,197.94 ENTER ↓ 143 ENTER ↓, $73,173.32 ENTER ↓ ↓, then hit IRR and CPT. The screen goes blank during the trial and error computations; then should show .506426.